

Multi-bump solutions for a class of quasilinear problems involving variable exponents *

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Abstract

We establish the existence of multi-bump solutions for the following class of quasilinear problems

$$-\Delta_{p(x)}u + (\lambda V(x) + Z(x))u^{p(x)-1} = f(x, u) \text{ in } \mathbb{R}^N, u \geq 0 \text{ in } \mathbb{R}^N,$$

where the nonlinearity $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function having a subcritical growth and potentials $V, Z: \mathbb{R}^N \rightarrow \mathbb{R}$ are continuous functions verifying some hypotheses. The main tool used is the variational method.

1 Introduction

In this paper, we considered the existence and multiplicity of solutions for the following class of problems

$$(P_\lambda) \quad \begin{cases} -\Delta_{p(x)}u + (\lambda V(x) + Z(x))u^{p(x)-1} = f(x, u), & \text{in } \mathbb{R}^N, \\ u \geq 0, & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases}$$

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where $\Delta_{p(x)}$ is the $p(x)$ -Laplacian operator given by

$$\Delta_{p(x)}u = \operatorname{div} \left(|\nabla u|^{p(x)-2} \nabla u \right).$$

Here, $\lambda > 0$ is a parameter, $p: \mathbb{R}^N \rightarrow \mathbb{R}$ is a Lipschitz function, $V, Z: \mathbb{R}^N \rightarrow \mathbb{R}$ are continuous functions with $V \geq 0$, and $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous having a subcritical growth. Furthermore, we take into account the following set of hypotheses:

$$(H_1) \quad 1 < p_- \leq p_+ < N.$$

$$(H_2) \quad \Omega = \operatorname{int} V^{-1}(0) \neq \emptyset \text{ and bounded, } \overline{\Omega} = V^{-1}(0) \text{ and } \Omega \text{ can be decomposed in } k \text{ connected components } \Omega_1, \dots, \Omega_k \text{ with } \operatorname{dist}(\Omega_i, \Omega_j) > 0, i \neq j.$$

$$(H_3) \quad \text{There exists } M > 0 \text{ such that}$$

$$\lambda V(x) + Z(x) \geq M, \forall x \in \mathbb{R}^N, \lambda \geq 1.$$

$$(H_4) \quad \text{There exists } K > 0 \text{ such that}$$

$$|Z(x)| \leq K, \forall x \in \mathbb{R}^N.$$

$$(f_1)$$

$$\limsup_{|t| \rightarrow \infty} \frac{|f(x, t)|}{|t|^{q(x)-1}} < \infty, \text{ uniformly in } x \in \mathbb{R}^N,$$

where $q: \mathbb{R}^N \rightarrow \mathbb{R}$ is continuous with $p_+ < q_-$ and $q \ll p^*$.

$$(f_2) \quad f(x, t) = o(|t|^{p_+-1}), t \rightarrow 0, \text{ uniformly in } x \in \mathbb{R}^N.$$

$$(f_3) \quad \text{There exists } \theta > p_+ \text{ such that}$$

$$0 < \theta F(x, t) \leq f(x, t)t, \forall x \in \mathbb{R}^N, t > 0,$$

where $F(x, t) = \int_0^t f(x, s) ds$.

$$(f_4) \quad \frac{f(x, t)}{t^{p_+-1}} \text{ is strictly increasing in } (0, \infty), \text{ for each } x \in \mathbb{R}^N.$$

$$(f_5) \quad \forall a, b \in \mathbb{R}, a < b, \sup_{\substack{x \in \mathbb{R}^N \\ t \in [a, b]}} |f(x, t)| < \infty.$$

A typical example of nonlinearity verifying $(f_1) - (f_5)$ is

$$f(x, t) = |t|^{q(x)-2}t, \forall x \in \mathbb{R}^N \text{ and } \forall t \in \mathbb{R},$$

where $p_+ < q_-$ and $q \ll p^*$.

Partial differential equations involving the $p(x)$ -Laplacian arise, for instance, as a mathematical model for problems involving electrorheological fluids and image restorations, see [1, 2, 11, 12, 13, 28]. This explains the intense research on this subject in the last decades. A lot of works, mainly treating nonlinearities with subcritical growth, are available (see [4, 5, 6, 9, 7, 8, 16, 17, 18, 20, 21, 22, 23, 27] for interesting works). Nevertheless, to the best of the author's knowledge, this is the first work dealing with multi-bump solutions for this class of problems.

The motivation to investigate problem (P_λ) in the setting of variable exponents has been the papers [3] and [15]. In [15], inspired by [14] and [29] the authors considered (P_λ) for $p = 2$ and $f(u) = u^q$, $q \in (1, \frac{N+2}{N-2})$ if $N \geq 3$; $q \in (1, \infty)$ if $N = 1, 2$. The authors showed that (P_λ) has at least $2^k - 1$ solutions u_λ for large values of λ . More precisely, one solution for each non-empty subset Υ of $\{1, \dots, k\}$. Moreover, fixed $\Upsilon \subset \{1, \dots, k\}$, it was proved that, for any sequence $\lambda_n \rightarrow \infty$ we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $W^{1,p(x)}(\mathbb{R}^N)$ to a function u , which satisfies $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

$$\begin{cases} -\Delta u + Z(x)u = u^q, & \text{in } \Omega_j, \\ u \in H_0^1(\Omega_j), u > 0, & \text{in } \Omega_j. \end{cases}$$

In [3], employing some different arguments than those used in [15], Alves extended the results described above to the p -Laplacian operator, assuming that in (P_λ) the nonlinearity f possesses a subcritical growth and $2 \leq p < N$. In particular, fixed $\Upsilon \subset \{1, \dots, k\}$, for any sequence $\lambda_n \rightarrow \infty$ we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $W^{1,p(x)}(\mathbb{R}^N)$ to a function u , which satisfies $u = 0$ outside Ω_Υ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

$$\begin{cases} -\Delta_p u + Z(x)u = f(u), & \text{in } \Omega_j, \\ u \in W_0^{1,p}(\Omega_j), u > 0, & \text{in } \Omega_j. \end{cases}$$

In the present paper, we extend the results found in [3] to the $p(x)$ -Laplacian operator. However, we would like emphasize that in a lot of estimates, we have used

different arguments from that found in [3]. The main difference is related to the fact that for equations involving the $p(x)$ -Laplacian operator it is not clear that Moser's iteration method is a good tool to get the estimates for the L^∞ -norm. Here, we adapt some ideas explored in [18] and [24] to get these estimates. For more details see Section 5.

Since we intend to find nonnegative solutions, throughout this paper, we replace f by $f^+ : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f^+(x, t) = \begin{cases} f(x, t), & \text{if } t > 0 \\ 0, & \text{if } t \leq 0. \end{cases}$$

Nevertheless, for the sake of simplicity, we still write f instead of f^+ .

The main theorem in this paper is the following:

Theorem 1.1 *Assume that $(H_1) - (H_4)$ and $(f_1) - (f_5)$ hold. Then, there exist $\lambda_0 > 0$ with the following property: for any non-empty subset Υ of $\{1, 2, \dots, k\}$ and $\lambda \geq \lambda_0$, problem (P_λ) has a solution u_λ . Moreover, if we fix the subset Υ , then for any sequence $\lambda_n \rightarrow \infty$ we can extract a subsequence (λ_{n_i}) such that $(u_{\lambda_{n_i}})$ converges strongly in $W^{1,p(x)}(\mathbb{R}^N)$ to a function u , which satisfies $u = 0$ outside $\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for*

$$\begin{cases} -\Delta_{p(x)} u + Z(x)u = f(x, u), & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j), u \geq 0, & \text{in } \Omega_j. \end{cases}$$

Notations: The following notations will be used in the present work:

- C and C_i will denote generic positive constant, which may vary from line to line;
- In all the integrals we omit the symbol dx .
- If u is a measurable function, we denote u^+ and u^- its positive and negative part, i.e., $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$.

- For $u, v \in C(\mathbb{R}^N)$, the notation $u \ll v$ means that $\inf_{x \in \mathbb{R}^N} (v(x) - u(x)) > 0$, $u_- = \inf_{x \in \mathbb{R}^N} u(x)$. Moreover, we will denote by u^* the function

$$u^*(x) = \begin{cases} \frac{Nu(x)}{N-u(x)}, & \text{if } u(x) < N, \\ \infty, & \text{if } u(x) \geq N. \end{cases}$$

2 Preliminaries on variable exponents Lebesgue and Sobolev spaces

In this section, we recall some results on variable exponents Lebesgue and Sobolev spaces found in [8, 19, 21] and their references.

Let $h \in L^\infty(\mathbb{R}^N)$ with $h_- = \text{ess inf}_{\mathbb{R}^N} h \geq 1$. The *variable exponent Lebesgue space* $L^{h(x)}(\mathbb{R}^N)$ is defined by

$$L^{h(x)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R}; u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{h(x)} < \infty \right\},$$

endowed with the norm

$$|u|_{h(x)} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left| \frac{u}{\lambda} \right|^{h(x)} \leq 1 \right\}.$$

The *variable exponent Sobolev space* is defined by

$$W^{1,h(x)}(\mathbb{R}^N) = \{ u \in L^{h(x)}(\mathbb{R}^N); |\nabla u| \in L^{h(x)}(\mathbb{R}^N) \},$$

with the norm

$$\|u\|_{1,h(x)} = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left(\left| \frac{\nabla u}{\lambda} \right|^{h(x)} + \left| \frac{u}{\lambda} \right|^{h(x)} \right) \leq 1 \right\}.$$

If $h_- > 1$, the spaces $L^{h(x)}(\mathbb{R}^N)$ and $W^{1,h(x)}(\mathbb{R}^N)$ are separable and reflexive with these norms.

We are mainly interested in subspaces of $W^{1,h(x)}(\mathbb{R}^N)$ given by

$$E_W = \left\{ u \in W^{1,h(x)}(\mathbb{R}^N); \int_{\mathbb{R}^N} W(x)|u|^{h(x)} < \infty \right\},$$

where $W \in C(\mathbb{R}^N)$ such that $W_- > 0$. Endowing E_W with the norm

$$\|u\|_W = \inf \left\{ \lambda > 0; \int_{\mathbb{R}^N} \left(\left| \frac{\nabla u}{\lambda} \right|^{h(x)} + W(x) \left| \frac{u}{\lambda} \right|^{h(x)} \right) \leq 1 \right\},$$

E_W is a Banach space. Moreover, it is easy to see that $E_W \hookrightarrow W^{1,h(x)}(\mathbb{R}^N)$ continuously. In addition, we can show that E_W is reflexive. For the reader's convenience, we recall some basic results.

Proposition 2.1 *The functional $\varrho: E_W \rightarrow \mathbb{R}$ defined by*

$$\varrho(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{h(x)} + W(x) |u|^{h(x)} \right), \quad (2.1)$$

has the following properties:

- (i) *If $\|u\|_W \geq 1$, then $\|u\|_W^{h_-} \leq \varrho(u) \leq \|u\|_W^{h_+}$.*
- (ii) *If $\|u\|_W \leq 1$, then $\|u\|_W^{h_+} \leq \varrho(u) \leq \|u\|_W^{h_-}$.*

In particular, for a sequence (u_n) in E_W ,

$$\begin{aligned} \|u_n\|_W \rightarrow 0 &\iff \varrho(u_n) \rightarrow 0, \text{ and,} \\ (u_n) \text{ is bounded in } E_W &\iff \varrho(u_n) \text{ is bounded in } \mathbb{R}. \end{aligned}$$

Remark 2.2 *For the functional $\varrho_{h(x)}: L^{h(x)}(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by*

$$\varrho_{h(x)}(u) = \int_{\mathbb{R}^N} |u|^{h(x)},$$

the same conclusion of Proposition 2.1 also holds.

Proposition 2.3 *Let $m \in L^\infty(\mathbb{R}^N)$ with $0 < m_- \leq m(x) \leq h(x)$ for a.e. $x \in \mathbb{R}^N$. If $u \in L^{h(x)}(\mathbb{R}^N)$, then $|u|^{m(x)} \in L^{\frac{h(x)}{m(x)}}(\mathbb{R}^N)$ and*

$$\left\| |u|^{m(x)} \right\|_{\frac{h(x)}{m(x)}} \leq \max \left\{ |u|_{h(x)}^{m_-}, |u|_{h(x)}^{m_+} \right\} \leq |u|_{h(x)}^{m_-} + |u|_{h(x)}^{m_+}.$$

Related to the Lebesgue space $L^{h(x)}(\mathbb{R}^N)$, we have the following generalized Hölder's inequality.

Proposition 2.4 (Hölder's inequality) *If $h_- > 1$, let $h': \mathbb{R}^N \rightarrow \mathbb{R}$ such that*

$$\frac{1}{h(x)} + \frac{1}{h'(x)} = 1 \text{ for a.e. } x \in \mathbb{R}^N.$$

Then, for any $u \in L^{h(x)}(\mathbb{R}^N)$ and $v \in L^{h'(x)}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |uv| dx \leq \left(\frac{1}{h_-} + \frac{1}{h'_-} \right) \|u\|_{h(x)} \|v\|_{h'(x)}.$$

We can define *variable exponent Lebesgue spaces with vector values*. We say $u = (u_1, \dots, u_L): \mathbb{R}^N \rightarrow \mathbb{R}^L \in L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ if, and only if, $u_i \in L^{h(x)}(\mathbb{R}^N)$, for $i = 1, \dots, L$. On $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$, we consider the norm $\|u\|_{L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)} = \sum_{i=1}^L \|u_i\|_{h(x)}$.

We state below lemmas of Brezis-Lieb type. The proof of the two first results follows the same arguments explored at [25], while the proof of the latter can be found at [8].

Proposition 2.5 (Brezis-Lieb lemma, first version) *Let (u_n) be a bounded sequence in $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ such that $u_n(x) \rightarrow u(x)$ for a.e. $x \in \mathbb{R}^N$. Then, $u \in L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ and*

$$\int_{\mathbb{R}^N} \left| |u_n|^{h(x)} - |u_n - u|^{h(x)} - |u|^{h(x)} \right| dx = o_n(1). \quad (2.2)$$

Proposition 2.6 (Brezis-Lieb lemma, second version) *Let (u_n) be a bounded sequence in $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ with $h_- > 1$ and $u_n(x) \rightarrow u(x)$ for a.e. $x \in \mathbb{R}^N$. Then*

$$u_n \rightharpoonup u \text{ in } L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L).$$

Proposition 2.7 (Brezis-Lieb lemma, third version) *Let (u_n) be a bounded sequence in $L^{h(x)}(\mathbb{R}^N, \mathbb{R}^L)$ with $h_- > 1$ and $u_n(x) \rightarrow u(x)$ for a.e. $x \in \mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} \left| |u_n|^{h(x)-2} u_n - |u_n - u|^{h(x)-2} (u_n - u) - |u|^{h(x)-2} u \right|^{h'(x)} dx = o_n(1), \quad (2.3)$$

To finish this section, we notice that for any open subset $\Omega \subset \mathbb{R}^N$, we can define of the same way the spaces $L^{h(x)}(\Omega)$ and $W^{1,h(x)}(\Omega)$. Moreover, all the above propositions hold for these spaces and, besides, we have the following embedding Theorem of Sobolev's type.

Proposition 2.8 ([21, Theorems 1.1, 1.3]) *Let $\Omega \subset \mathbb{R}^N$ an open domain with the cone property, $h: \overline{\Omega} \rightarrow \mathbb{R}$ satisfying $1 < h_- \leq h_+ < N$ and $m \in L_+^\infty(\Omega)$.*

- (i) *If h is Lipschitz continuous and $h \leq m \leq h^*$, the embedding $W^{1,h(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$ is continuous;*
- (ii) *If Ω is bounded, h is continuous and $m \ll h^*$, the embedding $W^{1,h(x)}(\Omega) \hookrightarrow L^{m(x)}(\Omega)$ is compact.*

3 An auxiliary problem

In this section, we work with an auxiliary problem adapting the ideas explored in del Pino & Felmer [14] (see also [3]).

We start noting that the energy functional $I_\lambda: E_\lambda \rightarrow \mathbb{R}$ associated with (P_λ) is given by

$$I_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x)) |u|^{p(x)} \right) - \int_{\mathbb{R}^N} F(x, u),$$

where $E_\lambda = (E, \|\cdot\|_\lambda)$ with

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x) |u|^{p(x)} < \infty \right\},$$

and

$$\|u\|_\lambda = \inf \left\{ \sigma > 0; \varrho_\lambda \left(\frac{u}{\sigma} \right) \leq 1 \right\},$$

being

$$\varrho_\lambda(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x)) |u|^{p(x)} \right).$$

Thus $E_\lambda \hookrightarrow W^{1,p(x)}(\mathbb{R}^N)$ continuously for $\lambda \geq 1$ and E_λ is compactly embedded in $L_{loc}^{h(x)}(\mathbb{R}^N)$, for all $1 \leq h \ll p^*$. In addition, we can show that E_λ is a reflexive space. Also, being $\mathcal{O} \subset \mathbb{R}^N$ an open set, from the relation

$$\varrho_{\lambda,\mathcal{O}}(u) = \int_{\mathcal{O}} \left(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x)) |u|^{p(x)} \right) \geq M \int_{\mathcal{O}} |u|^{p(x)} = M \varrho_{p(x),\mathcal{O}}(u), \quad (3.4)$$

for all $u \in E_\lambda$ with $\lambda \geq 1$, writing $M = (1 - \delta)^{-1}\nu$, for some $0 < \delta < 1$ and $\nu > 0$, we derive

$$\varrho_{\lambda,\mathcal{O}}(u) - \nu \varrho_{p(x),\mathcal{O}}(u) \geq \delta \varrho_{\lambda,\mathcal{O}}(u), \quad \forall u \in E_\lambda, \lambda \geq 1. \quad (3.5)$$

Remark 3.1 *From the above commentaries, in this work the parameter λ will be always bigger than or equal to 1.*

We recall that for any $\epsilon > 0$, the hypotheses (f_1) , (f_2) and (f_5) yield

$$f(x, t) \leq \epsilon |t|^{p(x)-1} + C_\epsilon |t|^{q(x)-1}, \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \quad (3.6)$$

and, consequently,

$$F(x, t) \leq \epsilon |t|^{p(x)} + C_\epsilon |t|^{q(x)}, \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \quad (3.7)$$

where C_ϵ depends on ϵ . Moreover, for each $\nu > 0$ fixed, the assumptions (f_2) and (f_3) allow us considering the function $a: \mathbb{R}^N \rightarrow \mathbb{R}$ given by

$$a(x) = \min \left\{ a > 0; \frac{f(x, a)}{a^{p(x)-1}} = \nu \right\}. \quad (3.8)$$

From (f_2) , it follows that

$$0 < a_- = \inf_{x \in \mathbb{R}^N} a(x). \quad (3.9)$$

Using the function $a(x)$, we set the function $\tilde{f}: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\tilde{f}(x, t) = \begin{cases} f(x, t), & t \leq a(x) \\ \nu t^{p(x)-1}, & t \geq a(x) \end{cases},$$

which fulfills the inequality

$$\tilde{f}(x, t) \leq \nu |t|^{p(x)-1}, \forall x \in \mathbb{R}^N, t \in \mathbb{R}. \quad (3.10)$$

Thus

$$\tilde{f}(x, t)t \leq \nu |t|^{p(x)}, \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \quad (3.11)$$

and

$$\tilde{F}(x, t) \leq \frac{\nu}{p(x)} |t|^{p(x)}, \forall x \in \mathbb{R}^N, t \in \mathbb{R}, \quad (3.12)$$

where $\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) ds$.

Now, once that $\Omega = \text{int } V^{-1}(0)$ is formed by k connected components $\Omega_1, \dots, \Omega_k$ with $\text{dist}(\Omega_i, \Omega_j) > 0$, $i \neq j$, then for each $j \in \{1, \dots, k\}$, we are able to fix a smooth bounded domain Ω'_j such that

$$\overline{\Omega'_j} \subset \Omega'_j \text{ and } \overline{\Omega'_i} \cap \overline{\Omega'_j} = \emptyset, \text{ for } i \neq j. \quad (3.13)$$

From now on, we fix a non-empty subset $\Upsilon \subset \{1, \dots, k\}$ and

$$\Omega_\Upsilon = \bigcup_{j \in \Upsilon} \Omega_j, \Omega'_\Upsilon = \bigcup_{j \in \Upsilon} \Omega'_j, \chi_\Upsilon = \begin{cases} 1, & \text{if } x \in \Omega'_\Upsilon \\ 0, & \text{if } x \notin \Omega'_\Upsilon. \end{cases}$$

Using the above notations, we set the functions

$$g(x, t) = \chi_\Upsilon(x)f(x, t) + (1 - \chi_\Upsilon(x))\tilde{f}(x, t), (x, t) \in \mathbb{R}^N \times \mathbb{R}$$

and

$$G(x, t) = \int_0^t g(x, s) ds, (x, t) \in \mathbb{R}^N \times \mathbb{R},$$

and the auxiliary problem

$$(A_\lambda) \begin{cases} -\Delta_{p(x)}u + (\lambda V(x) + Z(x))|u|^{p(x)-2}u = g(x, u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases}$$

The problem (A_λ) is related to (P_λ) , in the sense that, if u_λ is a solution for (A_λ) verifying

$$u_\lambda(x) \leq a(x), \forall x \in \mathbb{R}^N \setminus \Omega'_\Upsilon,$$

then it is a solution for (P_λ) .

In comparison to (P_λ) , problem (A_λ) has the advantage that the energy functional associated with (A_λ) , namely, $\phi_\lambda: E_\lambda \rightarrow \mathbb{R}$ given by

$$\phi_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x))|u|^{p(x)} \right) - \int_{\mathbb{R}^N} G(x, u),$$

satisfies the (PS) condition, whereas I_λ does not necessarily satisfy this condition. This way, the mountain pass level (see Theorem 3.6) is a critical value for ϕ_λ .

Proposition 3.2 *ϕ_λ satisfies the mountain pass geometry.*

Proof. From (3.7) and (3.12),

$$\phi_\lambda(u) \geq \frac{1}{p_+} \varrho_\lambda(u) - \epsilon \int_{\mathbb{R}^N} |u|^{p(x)} - C_\epsilon \int_{\mathbb{R}^N} |u|^{q(x)} - \frac{\nu}{p_-} \int_{\mathbb{R}^N} |u|^{p(x)},$$

for $\epsilon > 0$ and $C_\epsilon > 0$ be a constant depending on ϵ . By (3.4), fixing $\epsilon < \frac{M}{p_+}$ and $\nu < p_- M \left(\frac{1}{p_+} - \frac{\epsilon}{M} \right)$ and assuming $\|u\|_\lambda < \min \{1, 1/C_q\}$, where $|v|_{q(x)} \leq C_q \|v\|_\lambda$, $\forall v \in E_\lambda$, we derive from Proposition 2.1

$$\phi_\lambda(u) \geq \alpha \|u\|_\lambda^{p_+} - C \|u\|_\lambda^{q_-},$$

where $\alpha = \left(\frac{1}{p_+} - \frac{\epsilon}{M} \right) - \frac{\nu}{p_- M} > 0$. Once $p_+ < q_-$, the first part of the mountain pass geometry is satisfied. Now, fixing $v \in C_0^\infty(\Omega_\Upsilon)$, we have for $t \geq 0$

$$\phi_\lambda(tv) = \int_{\mathbb{R}^N} \frac{t^{p(x)}}{p(x)} \left(|\nabla v|^{p(x)} + Z(x) \right) |v|^{p(x)} - \int_{\mathbb{R}^N} F(x, tv).$$

If $t > 1$, by (f₃),

$$\phi_\lambda(tv) \leq \frac{t^{p_+}}{p_-} \int_{\mathbb{R}^N} \left(|\nabla v|^{p(x)} + Z(x) \right) |v|^{p(x)} - C_1 t^\theta \int_{\mathbb{R}^N} |v|^\theta - C_2,$$

and so,

$$\phi_\lambda(tv) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty.$$

The last limit implies that ϕ_λ verifies the second geometry of the mountain pass. ■

Proposition 3.3 *All $(PS)_d$ sequences for ϕ_λ are bounded in E_λ .*

Proof. Let (u_n) be a $(PS)_d$ sequence for ϕ_λ . So, there is $n_0 \in \mathbb{N}$ such that

$$\phi_\lambda(u_n) - \frac{1}{\theta} \phi'_\lambda(u_n) u_n \leq d + 1 + \|u_n\|_\lambda, \quad \text{for } n \geq n_0.$$

On the other hand, by (3.11) and (3.12)

$$\tilde{F}(x, t) - \frac{1}{\theta} \tilde{f}(x, t) t \leq \left(\frac{1}{p(x)} - \frac{1}{\theta} \right) \nu |t|^{p(x)}, \quad \forall x \in \mathbb{R}^N, t \in \mathbb{R},$$

which together with (3.5) gives

$$\phi_\lambda(u_n) - \frac{1}{\theta} \phi'_\lambda(u_n) u_n \geq \left(\frac{1}{p_+} - \frac{1}{\theta} \right) \delta \varrho_\lambda(u_n), \quad \forall n \in \mathbb{N}.$$

Hence

$$d + 1 + \max \left\{ \varrho_\lambda(u_n)^{1/p_-}, \varrho_\lambda(u_n)^{1/p_+} \right\} \geq \left(\frac{1}{p_+} - \frac{1}{\theta} \right) \delta \varrho_\lambda(u_n), \quad \forall n \geq n_0,$$

from where it follows that (u_n) is bounded in E_λ . ■

Proposition 3.4 *If (u_n) is a $(PS)_d$ sequence for ϕ_λ , then given $\epsilon > 0$, there is $R > 0$ such that*

$$\limsup_n \int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x)) |u_n|^{p(x)} \right) < \epsilon. \quad (3.14)$$

Hence, once that g has a subcritical growth, if $u \in E_\lambda$ is the weak limit of (u_n) , then

$$\int_{\mathbb{R}^N} g(x, u_n) u_n dx \rightarrow \int_{\mathbb{R}^N} g(x, u) u dx \quad \text{and} \quad \int_{\mathbb{R}^N} g(x, u_n) v dx \rightarrow \int_{\mathbb{R}^N} g(x, u) v dx, \quad \forall v \in E_\lambda.$$

Proof. Let (u_n) be a $(PS)_d$ sequence for ϕ_λ , $R > 0$ large such that $\Omega'_\Gamma \subset B_{\frac{R}{2}}(0)$ and $\eta_R \in C^\infty(\mathbb{R}^N)$ satisfying

$$\eta_R(x) = \begin{cases} 0, & x \in B_{\frac{R}{2}}(0) \\ 1, & x \in \mathbb{R}^N \setminus B_R(0) \end{cases},$$

$0 \leq \eta_R \leq 1$ and $|\nabla \eta_R| \leq \frac{C}{R}$, where $C > 0$ does not depend on R . This way,

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x)) |u_n|^{p(x)} \right) \eta_R \\ &= \phi'_\lambda(u_n) (u_n \eta_R) - \int_{\mathbb{R}^N} u_n |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \eta_R + \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} \tilde{f}(x, u_n) u_n \eta_R. \end{aligned}$$

Denoting

$$I = \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x)) |u_n|^{p(x)} \right) \eta_R,$$

it follows from (3.11),

$$I \leq \phi'_\lambda(u_n) (u_n \eta_R) + \frac{C}{R} \int_{\mathbb{R}^N} |u_n| |\nabla u_n|^{p(x)-1} + \nu \int_{\mathbb{R}^N} |u_n|^{p(x)} \eta_R.$$

Using Hölder's inequality 2.4 and Proposition 2.3, we derive

$$I \leq \phi'_\lambda(u_n) (u_n \eta_R) + \frac{C}{R} |u_n|_{p(x)} \max \left\{ |\nabla u_n|_{p(x)}^{p_- - 1}, |\nabla u_n|_{p(x)}^{p_+ - 1} \right\} + \frac{\nu}{M} I.$$

Since (u_n) and $(|\nabla u_n|)$ are bounded in $L^{p(x)}(\mathbb{R}^N)$ and $\frac{\nu}{M} = 1 - \delta$, we obtain

$$\int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x)) |u_n|^{p(x)} \right) \leq o_n(1) + \frac{C}{R}.$$

Therefore

$$\limsup_n \int_{\mathbb{R}^N \setminus B_R(0)} \left(|\nabla u_n|^{p(x)} + (\lambda V(x) + Z(x)) |u_n|^{p(x)} \right) \leq \frac{C}{R}.$$

So, given $\epsilon > 0$, choosing a $R > 0$ possibly still bigger, we have that $\frac{C}{R} < \epsilon$, which proves (3.14). Now, we will show that

$$\int_{\mathbb{R}^N} g(x, u_n) u_n \rightarrow \int_{\mathbb{R}^N} g(x, u) u.$$

Using the fact that $g(x, u)u \in L^1(\mathbb{R}^N)$ together with (3.14) and Sobolev embeddings, given $\epsilon > 0$, we can choose $R > 0$ such that

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N \setminus B_R(0)} |g(x, u_n) u_n| \leq \frac{\epsilon}{4} \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_R(0)} |g(x, u) u| \leq \frac{\epsilon}{4}.$$

On the other hand, since g has a subcritical growth, we have by compact embeddings

$$\int_{B_R(0)} g(x, u_n) u_n \rightarrow \int_{B_R(0)} g(x, u) u.$$

Combining the above informations, we conclude that

$$\int_{\mathbb{R}^N} g(x, u_n) u_n \rightarrow \int_{\mathbb{R}^N} g(x, u) u.$$

The same type of arguments works to prove that

$$\int_{\mathbb{R}^N} g(x, u_n) v \rightarrow \int_{\mathbb{R}^N} g(x, u) v \quad \forall v \in E_\lambda.$$

■

Proposition 3.5 ϕ_λ verifies the (PS) condition.

Proof. Let (u_n) be a $(PS)_d$ sequence for ϕ_λ and $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ . Thereby, by Proposition 3.4

$$\int_{\mathbb{R}^N} g(x, u_n) u_n \rightarrow \int_{\mathbb{R}^N} g(x, u) u \quad \text{and} \quad \int_{\mathbb{R}^N} g(x, u_n) v \rightarrow \int_{\mathbb{R}^N} g(x, u) v, \quad \forall v \in E_\lambda.$$

Moreover, the weak limit also give

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla (u_n - u) \rightarrow 0$$

and

$$\int_{\mathbb{R}^N} (\lambda V(x) + Z(x)) |u|^{p(x)-2} u (u_n - u) \rightarrow 0.$$

Now, if

$$P_n^1(x) = \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \cdot (\nabla u_n - \nabla u)$$

and

$$P_n^2(x) = (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) (u_n - u),$$

we derive

$$\begin{aligned} \int_{\mathbb{R}^N} \left(P_n^1(x) + (\lambda V(x) + Z(x)) P_n^2(x) \right) &= \phi'_\lambda(u_n) u_n + \int_{\mathbb{R}^N} g(x, u_n) u_n - \phi'_\lambda(u) u - \int_{\mathbb{R}^N} g(x, u) u \\ &\quad - \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \cdot \nabla (u_n - u) + (\lambda V(x) + Z(x)) |u|^{p(x)-2} u (u_n - u) \right). \end{aligned}$$

Recalling that $\phi'_\lambda(u_n) u_n = o_n(1)$ and $\phi'_\lambda(u) u = o_n(1)$, the above limits lead to

$$\int_{\mathbb{R}^N} \left(P_n^1(x) + (\lambda V(x) + Z(x)) P_n^2(x) \right) \rightarrow 0.$$

Now, the conclusion follows as in [8]. ■

Theorem 3.6 *The problem (A_λ) has a (nonnegative) solution, for all $\lambda \geq 1$.*

Proof. The proof is an immediate consequence of the Mountain Pass Theorem due to Ambrosetti & Rabinowitz [10]. ■

4 The $(PS)_\infty$ condition

A sequence $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$ is called a $(PS)_\infty$ sequence for the family $(\phi_\lambda)_{\lambda \geq 1}$, if there is a sequence $(\lambda_n) \subset [1, \infty)$ with $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, verifying

$$\phi_{\lambda_n}(u_n) \rightarrow c \text{ and } \|\phi'_{\lambda_n}(u_n)\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Proposition 4.1 *Let $(u_n) \subset W^{1,p(x)}(\mathbb{R}^N)$ be a $(PS)_\infty$ sequence for $(\phi_\lambda)_{\lambda \geq 1}$. Then, up to a subsequence, there exists $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$. Furthermore,*

(i) $\varrho_{\lambda_n}(u_n - u) \rightarrow 0$ and, consequently, $u_n \rightarrow u$ in $W^{1,p(x)}(\mathbb{R}^N)$;

(ii) $u = 0$ in $\mathbb{R}^N \setminus \Omega_\Upsilon$, $u \geq 0$ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a solution for

$$(P_j) \begin{cases} -\Delta_{p(x)} u + Z(x)|u|^{p(x)-2}u = f(x, u), & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j); \end{cases}$$

(iii) $\int_{\mathbb{R}^N} \lambda_n V(x) |u_n|^{p(x)} \rightarrow 0$;

(iv) $\varrho_{\lambda_n, \Omega'_j}(u_n) \rightarrow \int_{\Omega_j} (|\nabla u|^{p(x)} + Z(x)|u|^{p(x)})$, for $j \in \Upsilon$;

(v) $\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_n) \rightarrow 0$;

(vi) $\phi_{\lambda_n}(u_n) \rightarrow \int_{\Omega_\Upsilon} \frac{1}{p(x)} (|\nabla u|^{p(x)} + Z(x)|u|^{p(x)}) - \int_{\Omega_\Upsilon} F(x, u)$.

Proof. Using the same reasoning as in the proof of Proposition 3.3, we obtain that $(\varrho_{\lambda_n}(u_n))$ is bounded in \mathbb{R} . Then $(\|u_n\|_{\lambda_n})$ is bounded in \mathbb{R} and (u_n) is bounded in $W^{1,p(x)}(\mathbb{R}^N)$. So, up to a subsequence, there exists $u \in W^{1,p(x)}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ in } W^{1,p(x)}(\mathbb{R}^N) \text{ and } u_n(x) \rightarrow u(x) \text{ for a.e. } x \in \mathbb{R}^N.$$

Now, for each $m \in \mathbb{N}$, we define $C_m = \left\{ x \in \mathbb{R}^N; V(x) \geq \frac{1}{m} \right\}$. Without loss of generality, we can assume $\lambda_n < 2(\lambda_n - 1)$, $\forall n \in \mathbb{N}$. Thus

$$\int_{C_m} |u_n|^{p(x)} \leq \frac{2m}{\lambda_n} \int_{C_m} (\lambda_n V(x) + Z(x)) |u_n|^{p(x)} \leq \frac{2m}{\lambda_n} \varrho_{\lambda_n}(u_n) \leq \frac{C}{\lambda_n}.$$

By Fatou's lemma, we derive

$$\int_{C_m} |u|^{p(x)} = 0,$$

which implies that $u = 0$ in C_m and, consequently, $u = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$. From this, we are able to prove (i) – (vi).

(i) Since $u = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, repeating the argument explored in Proposition 3.5 we get

$$\int_{\mathbb{R}^N} \left(P_n^1(x) + (\lambda_n V(x) + Z(x)) P_n^2(x) \right) \rightarrow 0,$$

where

$$P_n^1(x) = \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) \cdot (\nabla u_n - \nabla u)$$

and

$$P_n^2(x) = (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) (u_n - u).$$

Therefore, $\varrho_{\lambda_n}(u_n - u) \rightarrow 0$, which implies $u_n \rightarrow u$ in $W^{1,p(x)}(\mathbb{R}^N)$.

(ii) Since $u \in W^{1,p(x)}(\mathbb{R}^N)$ and $u = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, we have $u \in W_0^{1,p(x)}(\Omega)$ or, equivalently, $u|_{\Omega_j} \in W_0^{1,p(x)}(\Omega_j)$, for $j = 1, \dots, k$. Moreover, the limit $u_n \rightarrow u$ in $W^{1,p(x)}(\mathbb{R}^N)$ combined with $\phi'_{\lambda_n}(u_n)\varphi \rightarrow 0$ for $\varphi \in C_0^\infty(\Omega_j)$ implies that

$$\int_{\Omega_j} \left(|\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi + Z(x) |u|^{p(x)-2} u \varphi \right) - \int_{\Omega_j} g(x, u) \varphi = 0, \quad (4.15)$$

showing that $u|_{\Omega_j}$ is a solution for

$$\begin{cases} -\Delta_{p(x)} u + Z(x) |u|^{p(x)-2} u = g(x, u), & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j). \end{cases}$$

This way, if $j \in \Upsilon$, then $u|_{\Omega_j}$ satisfies (P_j) . On the other hand, if $j \notin \Upsilon$, we must have

$$\int_{\Omega_j} \left(|\nabla u|^{p(x)} + Z(x) |u|^{p(x)} \right) - \int_{\Omega_j} \tilde{f}(x, u) u = 0.$$

The above equality combined with (3.11) and (3.5) gives

$$0 \geq \varrho_{\lambda, \Omega_j}(u) - \nu \varrho_{p(x), \Omega_j}(u) \geq \delta \varrho_{\lambda, \Omega_j}(u) \geq 0,$$

from where it follows $u|_{\Omega_j} = 0$. This proves $u = 0$ outside Ω_Υ and $u \geq 0$ in \mathbb{R}^N .

(iii) It follows from (i), since

$$\int_{\mathbb{R}^N} \lambda_n V(x) |u_n|^{p(x)} = \int_{\mathbb{R}^N} \lambda_n V(x) |u_n - u|^{p(x)} \leq 2 \varrho_{\lambda_n}(u_n - u).$$

(iv) Let $j \in \Upsilon$. From (i),

$$\varrho_{p(x), \Omega'_j}(u_n - u), \varrho_{p(x), \Omega'_j}(\nabla u_n - \nabla u) \rightarrow 0.$$

Then by Proposition 2.5,

$$\int_{\Omega'_j} (|\nabla u_n|^{p(x)} - |\nabla u|^{p(x)}) \rightarrow 0 \quad \text{and} \quad \int_{\Omega'_j} Z(x)(|u_n|^{p(x)} - |u|^{p(x)}) \rightarrow 0.$$

From (iii),

$$\int_{\Omega'_j} \lambda_n V(x)(|u_n|^{p(x)} - |u|^{p(x)}) = \int_{\Omega'_j \setminus \overline{\Omega_j}} \lambda_n V(x)|u_n|^{p(x)} \rightarrow 0.$$

This way

$$\varrho_{\lambda_n, \Omega'_j}(u_n) - \varrho_{\lambda_n, \Omega'_j}(u) \rightarrow 0.$$

Once $u = 0$ in $\Omega'_j \setminus \Omega_j$, we get

$$\varrho_{\lambda_n, \Omega'_j}(u_n) \rightarrow \int_{\Omega_j} (|\nabla u|^{p(x)} + Z(x)|u|^{p(x)}).$$

(v) By (i), $\varrho_{\lambda_n}(u_n - u) \rightarrow 0$, and so,

$$\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_n) \rightarrow 0.$$

(vi) We can write the functional ϕ_{λ_n} in the following way

$$\begin{aligned} \phi_{\lambda_n}(u_n) &= \sum_{j \in \Upsilon} \int_{\Omega'_j} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right) \\ &\quad + \int_{\mathbb{R}^N \setminus \Omega'_\Upsilon} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right) - \int_{\mathbb{R}^N} G(x, u_n). \end{aligned}$$

From (i) – (v),

$$\int_{\Omega'_j} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right) \rightarrow \int_{\Omega_j} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + Z(x)|u|^{p(x)} \right),$$

$$\int_{\mathbb{R}^N \setminus \Omega'_\Upsilon} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + (\lambda_n V(x) + Z(x))|u_n|^{p(x)} \right) \rightarrow 0.$$

and

$$\int_{\mathbb{R}^N} G(x, u_n) \rightarrow \int_{\Omega_\Upsilon} F(x, u).$$

Therefore

$$\phi_{\lambda_n}(u_n) \rightarrow \int_{\Omega_\Upsilon} \frac{1}{p(x)} (|\nabla u|^{p(x)} + Z(x)|u|^{p(x)}) - \int_{\Omega_\Upsilon} F(x, u).$$

■

5 The boundedness of the (A_λ) solutions

In this section, we study the boundedness outside Ω'_Υ for some solutions of (A_λ) . To this end, we adapt for our problem arguments found in [18] and [24].

Proposition 5.1 *Let (u_λ) be a family of solutions for (A_λ) such that $u_\lambda \rightarrow 0$ in $W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_\Upsilon)$, as $\lambda \rightarrow \infty$. Then, there exists $\lambda^* > 0$ with the following property:*

$$|u_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega'_\Upsilon} \leq a_-, \quad \forall \lambda \geq \lambda^*.$$

Hence, u_λ is a solution for (P_λ) for $\lambda \geq \lambda^*$.

Before to prove the above proposition, we need to show some technical lemmas.

Lemma 5.2 *There exist $x_1, \dots, x_l \in \partial\Omega'_\Upsilon$ and corresponding $\delta_{x_1}, \dots, \delta_{x_l} > 0$ such that*

$$\partial\Omega'_\Upsilon \subset \mathcal{N}(\partial\Omega'_\Upsilon) := \bigcup_{i=1}^l B_{\frac{\delta_{x_i}}{2}}(x_i).$$

Moreover,

$$q_+^{x_i} \leq (p_-^{x_i})^*, \quad (5.16)$$

where

$$q_+^{x_i} = \sup_{B_{\delta_{x_i}}(x_i)} q, \quad p_-^{x_i} = \inf_{B_{\delta_{x_i}}(x_i)} p \quad \text{and} \quad (p_-^{x_i})^* = \frac{Np_-^{x_i}}{N - p_-^{x_i}}.$$

Proof. From (3.13), $\overline{\Omega'_\Gamma} \subset \Omega'_\Gamma$. So, there is $\delta > 0$ such that

$$\overline{B_\delta(x)} \subset \mathbb{R}^N \setminus \overline{\Omega'_\Gamma}, \forall x \in \partial\Omega'_\Gamma.$$

Once $q \ll p^*$, there exists $\epsilon > 0$ such that $\epsilon \leq p^*(y) - q(y)$, for all $y \in \mathbb{R}^N$. Then, by continuity, for each $x \in \partial\Omega'_\Gamma$ we can choose a sufficiently small $0 < \delta_x \leq \delta$ such that

$$q_+^x \leq (p_-^x)^*,$$

where

$$q_+^x = \sup_{B_{\delta_x}(x)} q, \quad p_-^x = \inf_{B_{\delta_x}(x)} p \quad \text{and} \quad (p_-^x)^* = \frac{Np_-^x}{N - p_-^x}.$$

Covering $\partial\Omega'_\Gamma$ by the balls $B_{\frac{\delta_x}{2}}(x)$, $x \in \partial\Omega'_\Gamma$, and using its compactness, there are $x_1, \dots, x_l \in \partial\Omega'_\Gamma$ such that

$$\partial\Omega'_\Gamma \subset \bigcup_{i=1}^l B_{\frac{\delta_{x_i}}{2}}(x_i).$$

■

Lemma 5.3 *If u_λ is a solution for (A_λ) , in each $B_{\delta_{x_i}}(x_i)$, $i = 1, \dots, l$, given by Lemma 5.2, it is fulfilled*

$$\int_{A_{k, \bar{\delta}, x_i}} |\nabla u_\lambda|^{p_-^{x_i}} \leq C \left((k^{q_+} + 2) |A_{k, \tilde{\delta}, x_i}| + (\tilde{\delta} - \bar{\delta})^{-(p_-^{x_i})^*} \int_{A_{k, \tilde{\delta}, x_i}} (u_\lambda - k)^{(p_-^{x_i})^*} \right),$$

where $0 < \bar{\delta} < \tilde{\delta} < \delta_{x_i}$, $k \geq \frac{a_-}{4}$, $C = C(p_-, p_+, q_-, q_+, \nu, \delta_{x_i}) > 0$ is a constant independent of k , and for any $R > 0$, we denote by A_{k, R, x_i} the set

$$A_{k, R, x_i} = B_R(x_i) \cap \{x \in \mathbb{R}^N; u_\lambda(x) > k\}.$$

Proof. We choose arbitrarily $0 < \bar{\delta} < \tilde{\delta} < \delta_{x_i}$ and $\xi \in C^\infty(\mathbb{R}^N)$ with

$$0 \leq \xi \leq 1, \quad \text{supp } \xi \subset B_{\tilde{\delta}}(x_i), \quad \xi = 1 \text{ in } B_{\bar{\delta}}(x_i) \quad \text{and} \quad |\nabla \xi| \leq \frac{2}{\tilde{\delta} - \bar{\delta}}.$$

For $k \geq \frac{a_-}{4}$, we define $\eta = \xi^{p_+}(u_\lambda - k)^+$. We notice that

$$\nabla \eta = p_+ \xi^{p_+-1} (u_\lambda - k) \nabla \xi + \xi^{p_+} \nabla u_\lambda$$

on the set $\{u_\lambda > k\}$. Then, writing $u_\lambda = u$ and taking η as a test function, we obtain

$$\begin{aligned} p_+ \int_{A_{k, \tilde{\delta}, x_i}} \xi^{p_+-1} (u - k) |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \xi + \int_{A_{k, \tilde{\delta}, x_i}} \xi^{p_+} |\nabla u|^{p(x)} \\ + \int_{A_{k, \tilde{\delta}, x_i}} (\lambda V(x) + Z(x)) u^{p(x)-1} \xi^{p_+} (u - k) = \int_{A_{k, \tilde{\delta}, x_i}} g(x, u) \xi^{p_+} (u - k). \end{aligned}$$

If we set

$$J = \int_{A_{k, \tilde{\delta}, x_i}} \xi^{p_+} |\nabla u|^{p(x)},$$

using that $\nu \leq \lambda V(x) + Z(x)$, $\forall x \in \mathbb{R}^N$, we get

$$\begin{aligned} J \leq p_+ \int_{A_{k, \tilde{\delta}, x_i}} \xi^{p_+-1} (u - k) |\nabla u|^{p(x)-1} |\nabla \xi| \\ - \int_{A_{k, \tilde{\delta}, x_i}} \nu u^{p(x)-1} \xi^{p_+} (u - k) + \int_{A_{k, \tilde{\delta}, x_i}} g(x, u) \xi^{p_+} (u - k). \quad (5.17) \end{aligned}$$

From (5.17), (3.6) and (3.10),

$$\begin{aligned} J \leq p_+ \int_{A_{k, \tilde{\delta}, x_i}} \xi^{p_+-1} (u - k) |\nabla u|^{p(x)-1} |\nabla \xi| - \int_{A_{k, \tilde{\delta}, x_i}} \nu u^{p(x)-1} \xi^{p_+} (u - k) \\ + \int_{A_{k, \tilde{\delta}, x_i}} (\nu u^{p(x)-1} + C_\nu u^{q(x)-1}) \xi^{p_+} (u - k), \end{aligned}$$

from where it follows

$$J \leq p_+ \int_{A_{k, \tilde{\delta}, x_i}} \xi^{p_+-1} (u - k) |\nabla u|^{p(x)-1} |\nabla \xi| + C_\nu \int_{A_{k, \tilde{\delta}, x_i}} u^{q(x)-1} (u - k).$$

Using Young's inequality, we obtain, for $\chi \in (0, 1)$,

$$\begin{aligned} J \leq \frac{p_+(p_+-1)}{p_-} \chi^{\frac{p_-}{p_+-1}} J + \frac{2^{p_+} p_+}{p_-} \chi^{-p_+} \int_{A_{k, \tilde{\delta}, x_i}} \left(\frac{u - k}{\tilde{\delta} - \bar{\delta}} \right)^{p(x)} \\ + \frac{C_\nu (q_+ - 1)}{q_-} \int_{A_{k, \tilde{\delta}, x_i}} u^{q(x)} + \frac{C_\nu (1 + \delta_{x_i}^{q_+})}{q_-} \int_{A_{k, \tilde{\delta}, x_i}} \left(\frac{u - k}{\tilde{\delta} - \bar{\delta}} \right)^{q(x)}. \end{aligned}$$

Writing

$$Q = \int_{A_{k,\tilde{\delta},x_i}} \left(\frac{u-k}{\tilde{\delta}-\bar{\delta}} \right)^{(p_{-}^{x_i})^*},$$

for $\chi \approx 0^+$ fixed, due to (5.16), we deduce

$$\begin{aligned} J \leq \frac{1}{2}J + \frac{2^{p_+}p_+}{p_-}\chi^{-p_+}(|A_{k,\tilde{\delta},x_i}| + Q) + \frac{C_\nu 2^{q_+}(q_+ - 1)(1 + \delta_{x_i}^{q_+})}{q_-}(|A_{k,\tilde{\delta},x_i}| + Q) \\ + \frac{C_\nu 2^{q_+}(q_+ - 1)(1 + k^{q_+})}{q_-}|A_{k,\tilde{\delta},x_i}| + \frac{C_\nu(1 + \delta_{x_i}^{q_+})}{q_-}(|A_{k,\tilde{\delta},x_i}| + Q). \end{aligned}$$

Therefore

$$\int_{A_{k,\tilde{\delta},x_i}} |\nabla u|^{p(x)} \leq J \leq C \left[(k^{q_+} + 1)|A_{k,\tilde{\delta},x_i}| + Q \right],$$

for a positive constant $C = C(p_-, p_+, q_-, q_+, \nu, \delta_{x_i})$ which does not depend on k . Since

$$|\nabla u|^{p_{-}^{x_i}} - 1 \leq |\nabla u|^{p(x)}, \quad \forall x \in B_{\delta_{x_i}}(x_i),$$

we obtain

$$\begin{aligned} \int_{A_{k,\tilde{\delta},x_i}} |\nabla u|^{p_{-}^{x_i}} &\leq C \left[(k^{q_+} + 1)|A_{k,\tilde{\delta},x_i}| + Q \right] + |A_{k,\tilde{\delta},x_i}| \\ &\leq C \left((k^{q_+} + 2)|A_{k,\tilde{\delta},x_i}| + (\tilde{\delta} - \bar{\delta})^{-(p_{-}^{x_i})^*} \int_{A_{k,\tilde{\delta},x_i}} (u - k)^{(p_{-}^{x_i})^*} \right), \end{aligned}$$

for a positive constant $C = C(p_-, p_+, q_-, q_+, \nu, \delta_{x_i})$ which does not depend on k . ■

The next lemma can be found at ([26, Lemma 4.7]).

Lemma 5.4 *Let (J_n) be a sequence of nonnegative numbers satisfying*

$$J_{n+1} \leq CB^n J_n^{1+\eta}, \quad n = 0, 1, 2, \dots,$$

where $C, \eta > 0$ and $B > 1$. If

$$J_0 \leq C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}},$$

then $J_n \rightarrow 0$, as $n \rightarrow \infty$.

Lemma 5.5 *Let (u_λ) be a family of solutions for (A_λ) such that $u_\lambda \rightarrow 0$ in $W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_\Upsilon)$, as $\lambda \rightarrow \infty$. Then, there exists $\lambda^* > 0$ with the following property:*

$$|u_\lambda|_{\infty, \mathcal{N}(\partial\Omega'_\Upsilon)} \leq a_-, \forall \lambda \geq \lambda^*.$$

Proof. It is enough to prove the inequality in each ball $B_{\frac{\delta_{x_i}}{2}}(x_i)$, $i = 1, \dots, l$, given by Lemma 5.2. We set

$$\tilde{\delta}_n = \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2^{n+1}}, \quad \bar{\delta}_n = \frac{\tilde{\delta}_n + \tilde{\delta}_{n+1}}{2}, \quad k_n = \frac{a_-}{2} \left(1 - \frac{1}{2^{n+1}}\right), \quad \forall n = 0, 1, 2, \dots$$

Then

$$\tilde{\delta}_n \downarrow \frac{\delta_{x_i}}{2}, \quad \tilde{\delta}_{n+1} < \bar{\delta}_n < \tilde{\delta}_n, \quad k_n \uparrow \frac{a_-}{2}.$$

From now on, we fix

$$J_n(\lambda) = \int_{A_{k_n, \bar{\delta}_n, x_i}} (u_\lambda(x) - k_n)^{(p_-^{x_i})^*}, \quad n = 0, 1, 2, \dots$$

and $\xi \in C^1(\mathbb{R})$ such that

$$0 \leq \xi \leq 1, \quad \xi(t) = 1, \text{ for } t \leq \frac{1}{2}, \text{ and } \xi(t) = 0, \text{ for } t \geq \frac{3}{4}.$$

Setting

$$\xi_n(x) = \xi\left(\frac{2^{n+1}}{\delta_{x_i}}\left(|x - x_i| - \frac{\delta_{x_i}}{2}\right)\right), \quad x \in \mathbb{R}^N, \quad n = 0, 1, 2, \dots,$$

we have $\xi_n = 1$ in $B_{\tilde{\delta}_{n+1}}(x_i)$ and $\xi_n = 0$ outside $B_{\bar{\delta}_n}(x_i)$. Writing $u_\lambda = u$, we get

$$\begin{aligned} J_{n+1} &\leq \int_{A_{k_{n+1}, \bar{\delta}_n, x_i}} ((u(x) - k_{n+1})\xi_n(x))^{(p_-^{x_i})^*} \\ &= \int_{B_{\delta_{x_i}}(x_i)} ((u - k_{n+1})^+(x)\xi_n(x))^{(p_-^{x_i})^*} \\ &\leq C(N, p_-^{x_i}) \left(\int_{B_{\delta_{x_i}}(x_i)} |\nabla((u - k_{n+1})^+\xi_n)(x)|^{p_-^{x_i}} \right)^{\frac{(p_-^{x_i})^*}{p_-^{x_i}}} \\ &\leq C(N, p_-^{x_i}) \left(\int_{A_{k_{n+1}, \bar{\delta}_n, x_i}} |\nabla u|^{p_-^{x_i}} + \int_{A_{k_{n+1}, \bar{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} |\nabla \xi_n|^{p_-^{x_i}} \right)^{\frac{(p_-^{x_i})^*}{p_-^{x_i}}}. \end{aligned}$$

Since

$$|\nabla \xi_n(x)| \leq C(\delta_{x_i}) 2^{n+1}, \quad \forall x \in \mathbb{R}^N,$$

writing $J_{n+1}^{\frac{p_-^{x_i}}{(p_-^{x_i})^*}} = \tilde{J}_{n+1}$, we obtain

$$\tilde{J}_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}) \left(\int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} |\nabla u|^{p_-^{x_i}} + 2^{np_-^{x_i}} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \right).$$

Using Lemma 5.3,

$$\begin{aligned} \tilde{J}_{n+1} &\leq C(N, p_-^{x_i}, \delta_{x_i}) \left((k_{n+1}^{q_+} + 2) |A_{k_{n+1}, \tilde{\delta}_n, x_i}| \right. \\ &\quad \left. + \left(\frac{2^{n+3}}{\delta_{x_i}} \right)^{(p_-^{x_i})^*} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{(p_-^{x_i})^*} + 2^{np_-^{x_i}} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \right) \\ &\leq C(N, p_-^{x_i}, \delta_{x_i}) \left((k_{n+1}^{q_+} + 2) |A_{k_{n+1}, \tilde{\delta}_n, x_i}| \right. \\ &\quad \left. + 2^{n(p_-^{x_i})^*} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{(p_-^{x_i})^*} + 2^{np_-^{x_i}} \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \right). \end{aligned}$$

From Young's inequality

$$\int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{p_-^{x_i}} \leq C(p_-^{x_i}) \left(|A_{k_{n+1}, \tilde{\delta}_n, x_i}| + \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_{n+1})^{(p_-^{x_i})^*} \right).$$

Thus

$$\tilde{J}_{n+1} \leq C(N, p_-^{x_i}, \delta_{x_i}) \left(\left(\left(\frac{a_-}{2} \right)^{q_+} + 2 + 2^{np_-^{x_i}} \right) |A_{k_{n+1}, \tilde{\delta}_n, x_i}| + 2^{n(p_-^{x_i})^*} J_n + 2^{np_-^{x_i}} J_n \right).$$

Now, since

$$J_n \geq \int_{A_{k_{n+1}, \tilde{\delta}_n, x_i}} (u - k_n)^{(p_-^{x_i})^*} \geq (k_{n+1} - k_n)^{(p_-^{x_i})^*} |A_{k_{n+1}, \tilde{\delta}_n, x_i}|$$

it follows that

$$|A_{k_{n+1}, \tilde{\delta}_n, x_i}| \leq \left(\frac{2^{n+3}}{a_-} \right)^{(p_-^{x_i})^*} J_n,$$

and so,

$$\tilde{J}_{n+1} \leq C \left(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+ \right) \left(2^{n(p_-^{x_i})^*} J_n + 2^{n(p_-^{x_i} + (p_-^{x_i})^*)} J_n + 2^{n(p_-^{x_i})^*} J_n + 2^{np_-^{x_i}} J_n \right).$$

Fixing $\alpha = (p_-^{x_i} + (p_-^{x_i})^*)$, it follows that

$$J_{n+1} \leq C \left(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+ \right) \left(2^{\alpha \frac{(p_-^{x_i})^*}{p_-^{x_i}}} \right)^n J_n^{\frac{(p_-^{x_i})^*}{p_-^{x_i}}},$$

and consequently

$$J_{n+1} \leq C B^n J_n^{1+\eta},$$

where $C = C \left(N, p_-^{x_i}, \delta_{x_i}, a_-, q_+ \right)$, $B = 2^{\alpha \frac{(p_-^{x_i})^*}{p_-^{x_i}}}$ and $\eta = \frac{(p_-^{x_i})^*}{p_-^{x_i}} - 1$. Now, once that $u_\lambda \rightarrow 0$ in $W^{1,p(x)}(\mathbb{R}^N \setminus \Omega_\Gamma)$, as $\lambda \rightarrow \infty$, there exists $\lambda_i > 0$ such that

$$\int_{A_{\frac{a_-}{4}, \delta_{x_i}, x_i}} \left(u_\lambda - \frac{a_-}{4} \right)^{(p_-^{x_i})^*} = J_0(\lambda) \leq C^{-\frac{1}{\eta}} B^{-\frac{1}{\eta^2}}, \quad \lambda \geq \lambda_i.$$

From Lemma 5.4, $J_n(\lambda) \rightarrow 0$, $n \rightarrow \infty$, for all $\lambda \geq \lambda_i$, and so,

$$u_\lambda \leq \frac{a_-}{2} < a_-, \text{ in } B_{\frac{\delta_{x_i}}{2}}, \text{ for all } \lambda \geq \lambda_i.$$

Now, taking $\lambda^* = \max\{\lambda_1, \dots, \lambda_l\}$, we conclude that

$$|u_\lambda|_{\infty, \mathcal{N}(\partial\Omega'_\Gamma)} < a_-, \quad \forall \lambda \geq \lambda^*.$$

■

Proof of Proposition 5.1. Fix $\lambda \geq \lambda^*$, where λ^* is given at Lemma 5.5, and define $\tilde{u}_\lambda: \mathbb{R}^N \setminus \Omega'_\Gamma \rightarrow \mathbb{R}$ given by

$$\tilde{u}_\lambda(x) = (u_\lambda - a_-)^+(x).$$

From Lemma 5.5, $\tilde{u}_\lambda \in W_0^{1,p(x)}(\mathbb{R}^N \setminus \Omega'_\Gamma)$. Our goal is showing that $\tilde{u}_\lambda = 0$ in $\mathbb{R}^N \setminus \Omega'_\Gamma$. This implies

$$|u_\lambda|_{\infty, \mathbb{R}^N \setminus \Omega'_\Gamma} \leq a_-.$$

In fact, extending $\tilde{u}_\lambda = 0$ in Ω'_Γ and taking \tilde{u}_λ as a test function, we obtain

$$\int_{\mathbb{R}^N \setminus \Omega'_\Gamma} |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda \cdot \nabla \tilde{u}_\lambda + \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} (\lambda V(x) + Z(x)) u_\lambda^{p(x)-2} u_\lambda \tilde{u}_\lambda = \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} g(x, u_\lambda) \tilde{u}_\lambda.$$

Since

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} |\nabla u_\lambda|^{p(x)-2} \nabla u_\lambda \cdot \nabla \tilde{u}_\lambda &= \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} |\nabla \tilde{u}_\lambda|^{p(x)}, \\ \int_{\mathbb{R}^N \setminus \Omega'_\Gamma} (\lambda V(x) + Z(x)) u_\lambda^{p(x)-2} u_\lambda \tilde{u}_\lambda &= \int_{(\mathbb{R}^N \setminus \Omega'_\Gamma)_+} (\lambda V(x) + Z(x)) u_\lambda^{p(x)-2} (\tilde{u}_\lambda + a_-) \tilde{u}_\lambda \end{aligned}$$

and

$$\int_{\mathbb{R}^N \setminus \Omega'_\Gamma} g(x, u_\lambda) \tilde{u}_\lambda = \int_{(\mathbb{R}^N \setminus \Omega'_\Gamma)_+} \frac{g(x, u_\lambda)}{u_\lambda} (\tilde{u}_\lambda + a_-) \tilde{u}_\lambda,$$

where

$$(\mathbb{R}^N \setminus \Omega'_\Gamma)_+ = \{x \in \mathbb{R}^N \setminus \Omega'_\Gamma; u_\lambda(x) > 0\},$$

we derive

$$\int_{\mathbb{R}^N \setminus \Omega'_\Gamma} |\nabla \tilde{u}_\lambda|^{p(x)} + \int_{(\mathbb{R}^N \setminus \Omega'_\Gamma)_+} \left((\lambda V(x) + Z(x)) u_\lambda^{p(x)-2} - \frac{g(x, u_\lambda)}{u_\lambda} \right) (\tilde{u}_\lambda + a_-) \tilde{u}_\lambda = 0,$$

Now, by (3.10),

$$(\lambda V(x) + Z(x)) u_\lambda^{p(x)-2} - \frac{g(x, u_\lambda)}{u_\lambda} > \nu u_\lambda^{p(x)-2} - \frac{\tilde{f}(x, u_\lambda)}{u_\lambda} \geq 0 \quad \text{in} \quad (\mathbb{R}^N \setminus \Omega'_\Gamma)_+.$$

This form, $\tilde{u}_\lambda = 0$ in $(\mathbb{R}^N \setminus \Omega'_\Gamma)_+$. Obviously, $\tilde{u}_\lambda = 0$ at the points where $u_\lambda = 0$, consequently, $\tilde{u}_\lambda = 0$ in $\mathbb{R}^N \setminus \Omega'_\Gamma$. \blacksquare

6 A special critical value for ϕ_λ

For each $j = 1, \dots, k$, consider

$$I_j(u) = \int_{\Omega_j} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + Z(x) |u|^{p(x)} \right) - \int_{\Omega_j} F(x, u), \quad u \in W_0^{1,p(x)}(\Omega_j),$$

the energy functional associated to (P_j) , and

$$\phi_{\lambda,j}(u) = \int_{\Omega'_j} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + (\lambda V(x) + Z(x)) |u|^{p(x)} \right) - \int_{\Omega'_j} F(x, u), \quad u \in W^{1,p(x)}(\Omega'_j),$$

the energy functional associated to

$$\begin{cases} -\Delta_{p(x)}u + (\lambda V(x) + Z(x))|u|^{p(x)-2}u = f(x, u), & \text{in } \Omega'_j, \\ \frac{\partial u}{\partial \eta} = 0, & \text{on } \partial\Omega'_j. \end{cases}$$

It is fulfilled that I_j and $\phi_{\lambda,j}$ satisfy the mountain pass geometry and let

$$c_j = \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} I_j(\gamma(t)) \quad \text{and} \quad c_{\lambda,j} = \inf_{\gamma \in \Gamma_{\lambda,j}} \max_{t \in [0,1]} \phi_{\lambda,j}(\gamma(t)),$$

their respective mountain pass levels, where

$$\Gamma_j = \left\{ \gamma \in C\left([0, 1], W_0^{1,p(x)}(\Omega_j)\right) ; \gamma(0) = 0 \text{ and } I_j(\gamma(1)) < 0 \right\}$$

and

$$\Gamma_{\lambda,j} = \left\{ \gamma \in C\left([0, 1], W^{1,p(x)}(\Omega'_j)\right) ; \gamma(0) = 0 \text{ and } \phi_{\lambda,j}(\gamma(1)) < 0 \right\}.$$

Invoking the (PS) condition on I_j and $\phi_{\lambda,j}$, we ensure that there exist $w_j \in W_0^{1,p(x)}(\Omega_j)$ and $w_{\lambda,j} \in W^{1,p(x)}(\Omega'_j)$ such that

$$I_j(w_j) = c_j \quad \text{and} \quad I'_j(w_j) = 0$$

and

$$\phi_{\lambda,j}(w_{\lambda,j}) = c_{\lambda,j} \quad \text{and} \quad \phi'_{\lambda,j}(w_{\lambda,j}) = 0.$$

Lemma 6.1 *There holds that*

$$(i) \quad 0 < c_{\lambda,j} \leq c_j, \quad \forall \lambda \geq 1, \quad \forall j \in \{1, \dots, k\};$$

$$(ii) \quad c_{\lambda,j} \rightarrow c_j, \quad \text{as } \lambda \rightarrow \infty, \quad \forall j \in \{1, \dots, k\}.$$

Proof.

(i) Once $W_0^{1,p(x)}(\Omega_j) \subset W^{1,p(x)}(\Omega'_j)$ and $\phi_{\lambda,j}(\gamma(1)) = I_j(\gamma(1))$ for $\gamma \in \Gamma_j$, we have $\Gamma_j \subset \Gamma_{\lambda,j}$. This way

$$c_{\lambda,j} = \inf_{\gamma \in \Gamma_{\lambda,j}} \max_{t \in [0,1]} \phi_{\lambda,j}(\gamma(t)) \leq \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} \phi_{\lambda,j}(\gamma(t)) = \inf_{\gamma \in \Gamma_j} \max_{t \in [0,1]} I_j(\gamma(t)) = c_j.$$

- (ii) It suffices to show that $c_{\lambda_n,j} \rightarrow c_j$, as $n \rightarrow \infty$, for all sequences (λ_n) in $[1, \infty)$ with $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$. Let (λ_n) be such a sequence and consider an arbitrary subsequence of $(c_{\lambda_n,j})$ (not relabelled). Let $w_n \in W^{1,p(x)}(\Omega'_j)$ with

$$\phi_{\lambda_n,j}(w_n) = c_{\lambda_n,j} \text{ and } \phi'_{\lambda_n,j}(w_n) = 0.$$

By the previous item, $(c_{\lambda_n,j})$ is bounded. Then, there exists (w_{n_k}) subsequence of (w_n) such that $\phi_{\lambda_{n_k},j}(w_{n_k})$ converges and $\phi'_{\lambda_{n_k},j}(w_{n_k}) = 0$. Now, repeating the same type of arguments explored in the proof of Proposition 4.1, there is $w \in W_0^{1,p(x)}(\Omega_j) \setminus \{0\} \subset W^{1,p(x)}(\Omega'_j)$ such that

$$w_{n_k} \rightarrow w \text{ in } W^{1,p(x)}(\Omega'_j), \text{ as } k \rightarrow \infty.$$

Furthermore, we also can prove that

$$c_{\lambda_{n_k},j} = \phi_{\lambda_{n_k},j}(w_{n_k}) \rightarrow I_j(w)$$

and

$$0 = \phi'_{\lambda_{n_k},j}(w_{n_k}) \rightarrow I'_j(w).$$

Then, by (f_4) ,

$$\lim_k c_{\lambda_{n_k},j} \geq c_j.$$

The last inequality together with item (i) implies

$$c_{\lambda_{n_k},j} \rightarrow c_j, \text{ as } k \rightarrow \infty.$$

This establishes the asserted result. ■

In the sequel, let $R > 1$ verifying

$$0 < I_j\left(\frac{1}{R}w_j\right), I_j(Rw_j) < c_j, \text{ for } j = 1, \dots, k. \quad (6.18)$$

There holds that

$$c_j = \max_{t \in [1/R^2, 1]} I_j(tRw_j), \text{ for } j = 1, \dots, k.$$

Moreover, to simplify the notation, we rename the components Ω_j of Ω in way such that $\Upsilon = \{1, 2, \dots, l\}$ for some $1 \leq l \leq k$. Then, we define:

$$\gamma_0(t_1, \dots, t_l)(x) = \sum_{j=1}^l t_j R w_j(x), \quad \forall (t_1, \dots, t_l) \in [1/R^2, 1]^l,$$

$$\Gamma_* = \left\{ \gamma \in C([1/R^2, 1]^l, E_\lambda \setminus \{0\}) ; \gamma = \gamma_0 \text{ on } \partial[1/R^2, 1]^l \right\}$$

and

$$b_{\lambda, \Upsilon} = \inf_{\gamma \in \Gamma_*} \max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} \phi_\lambda(\gamma(t_1, \dots, t_l)).$$

Next, our intention is proving that $b_{\lambda, \Upsilon}$ is a critical value for ϕ_λ . However, to do this, we need to some technical lemmas. The arguments used are the same found in [3], however for reader's convenience we will repeat their proofs

Lemma 6.2 *For all $\gamma \in \Gamma_*$, there exists $(s_1, \dots, s_l) \in [1/R^2, 1]^l$ such that*

$$\phi'_{\lambda, j}(\gamma(s_1, \dots, s_l))(\gamma(s_1, \dots, s_l)) = 0, \quad \forall j \in \Upsilon.$$

Proof. Given $\gamma \in \Gamma_*$, consider $\tilde{\gamma}: [1/R^2, 1]^l \rightarrow \mathbb{R}^l$ such that

$$\tilde{\gamma}(\mathbf{t}) = \left(\phi'_{\lambda, 1}(\gamma(\mathbf{t}))\gamma(\mathbf{t}), \dots, \phi'_{\lambda, l}(\gamma(\mathbf{t}))\gamma(\mathbf{t}) \right), \quad \text{where } \mathbf{t} = (t_1, \dots, t_l).$$

For $\mathbf{t} \in \partial[1/R^2, 1]^l$, it holds $\tilde{\gamma}(\mathbf{t}) = \tilde{\gamma}_0(\mathbf{t})$. From this, we observe that there is no $\mathbf{t} \in \partial[1/R^2, 1]^l$ with $\tilde{\gamma}(\mathbf{t}) = 0$. Indeed, for any $j \in \Upsilon$,

$$\phi'_{\lambda, j}(\gamma_0(\mathbf{t}))\gamma_0(\mathbf{t}) = I'_j(t_j R w_j)(t_j R w_j).$$

This form, if $\mathbf{t} \in \partial[1/R^2, 1]^l$, then $t_{j_0} = 1$ or $t_{j_0} = \frac{1}{R^2}$, for some $j_0 \in \Upsilon$. Consequently,

$$\phi'_{\lambda, j_0}(\gamma_0(\mathbf{t}))\gamma_0(\mathbf{t}) = I'_{j_0}(R w_{j_0})(R w_{j_0}) \quad \text{or} \quad \phi'_{\lambda, j_0}(\gamma_0(\mathbf{t}))\gamma_0(\mathbf{t}) = I'_{j_0}\left(\frac{1}{R} w_{j_0}\right)\left(\frac{1}{R} w_{j_0}\right).$$

Therefore, if $\phi'_{\lambda, j_0}(\gamma_0(\mathbf{t}))\gamma_0(\mathbf{t}) = 0$, we get $I_{j_0}(R w_{j_0}) \geq c_{j_0}$ or $I_{j_0}\left(\frac{1}{R} w_{j_0}\right) \geq c_{j_0}$, which is a contradiction with (6.18).

Now, we compute the degree $\deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, \dots, 0))$. Since

$$\deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, \dots, 0)) = \deg(\tilde{\gamma}_0, (1/R^2, 1)^l, (0, \dots, 0)),$$

and, for $\mathbf{t} \in (1/R^2, 1)^l$,

$$\tilde{\gamma}_0(\mathbf{t}) = 0 \iff \mathbf{t} = \left(\frac{1}{R}, \dots, \frac{1}{R} \right),$$

we derive

$$\deg(\tilde{\gamma}, (1/R^2, 1)^l, (0, \dots, 0)) \neq 0.$$

This shows what was stated. ■

Proposition 6.3 *If $c_{\lambda, \Upsilon} = \sum_{j=1}^l c_{\lambda, j}$ and $c_{\Upsilon} = \sum_{j=1}^l c_j$, then*

- (i) $c_{\lambda, \Upsilon} \leq b_{\lambda, \Upsilon} \leq c_{\Upsilon}$, $\forall \lambda \geq 1$;
- (ii) $b_{\lambda, \Upsilon} \rightarrow c_{\Upsilon}$, as $\lambda \rightarrow \infty$;
- (iii) $\phi_{\lambda}(\gamma(\mathbf{t})) < c_{\Upsilon}$, $\forall \lambda \geq 1, \gamma \in \Gamma_*$ and $\mathbf{t} = (t_1, \dots, t_l) \in \partial[1/R^2, 1]^l$.

Proof.

(i) Once $\gamma_0 \in \Gamma_*$,

$$b_{\lambda, \Upsilon} \leq \max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} \phi_{\lambda}(\gamma_0(t_1, \dots, t_l)) = \max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} \sum_{j=1}^l I_j(t_j R w_j) = c_{\Upsilon}.$$

Now, fixing $\mathbf{s} = (s_1, \dots, s_l) \in [1/R^2, 1]^l$ given in Lemma 6.2 and recalling that

$$c_{\lambda, j} = \inf \left\{ \phi_{\lambda, j}(u) ; u \in W^{1, p(x)}(\Omega'_j) \setminus \{0\} \text{ and } \phi'_{\lambda, j}(u)u = 0 \right\},$$

it follows that

$$\phi_{\lambda, j}(\gamma(\mathbf{s})) \geq c_{\lambda, j}, \forall j \in \Upsilon.$$

From (3.12),

$$\phi_{\lambda, \mathbb{R}^N \setminus \Omega'_{\Upsilon}}(u) \geq 0, \forall u \in W^{1, p(x)}(\mathbb{R}^N \setminus \Omega'_{\Upsilon}),$$

which leads to

$$\phi_{\lambda}(\gamma(\mathbf{t})) \geq \sum_{j=1}^l \phi_{\lambda, j}(\gamma(\mathbf{t})), \forall \mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l.$$

Thus

$$\max_{(t_1, \dots, t_l) \in [1/R^2, 1]^l} \phi_{\lambda}(\gamma(t_1, \dots, t_l)) \geq \phi_{\lambda}(\gamma(\mathbf{s})) \geq c_{\lambda, \Upsilon},$$

showing that

$$b_{\lambda, \Upsilon} \geq c_{\lambda, \Upsilon};$$

- (ii) This limit is clear by the previous item, since we already know $c_{\lambda,j} \rightarrow c_j$, as $\lambda \rightarrow \infty$;
- (iii) For $\mathbf{t} = (t_1, \dots, t_l) \in \partial[1/R^2, 1]^l$, it holds $\gamma(\mathbf{t}) = \gamma_0(\mathbf{t})$. From this,

$$\phi_\lambda(\gamma(\mathbf{t})) = \sum_{j=1}^l I_j(t_j R w_j).$$

Writing

$$\phi_\lambda(\gamma(\mathbf{t})) = \sum_{\substack{j=1 \\ j \neq j_0}}^l I_j(t_j R w_j) + I_{j_0}(t_{j_0} R w_{j_0}),$$

where $t_{j_0} \in \{\frac{1}{R^2}, 1\}$, from (6.18) we derive

$$\phi_\lambda(\gamma(\mathbf{t})) \leq c_\Upsilon - \epsilon,$$

for some $\epsilon > 0$, so (iii). ■

Corollary 6.4 $b_{\lambda,\Upsilon}$ is a critical value of ϕ_λ , for λ sufficiently large.

Proof. Assume $b_{\tilde{\lambda},\Upsilon}$ is not a critical value of $\phi_{\tilde{\lambda}}$ for some $\tilde{\lambda}$. We will prove that exists λ_1 such that $\tilde{\lambda} < \lambda_1$. Indeed, by item (iii) of Proposition 6.3, we have seen that

$$\phi_\lambda(\gamma_0(\mathbf{t})) < c_\Upsilon, \forall \lambda \geq 1, \mathbf{t} \in \partial[1/R^2, 1]^l.$$

This way

$$\mathcal{M} = \max_{\mathbf{t} \in \partial[1/R^2, 1]^l} \phi_{\tilde{\lambda}}(\gamma_0(\mathbf{t})) < c_\Upsilon.$$

Since $b_{\lambda,\Upsilon} \rightarrow c_\Upsilon$ (item (ii) of Proposition 6.3), there exists $\lambda_1 > 1$ such that if $\lambda \geq \lambda_1$, then

$$\mathcal{M} < b_{\lambda,\Upsilon}.$$

So, if $\tilde{\lambda} \geq \lambda_1$, we can find $\tau = \tau(\tilde{\lambda}) > 0$ small enough, with the ensuing property

$$\mathcal{M} < b_{\tilde{\lambda},\Upsilon} - 2\tau. \tag{6.19}$$

From the deformation's lemma [30, Page 38], there is $\eta: E_\lambda \rightarrow E_\lambda$ such that

$$\eta\left(\phi_{\tilde{\lambda}}^{b_{\tilde{\lambda},\Upsilon} + \tau}\right) \subset \phi_{\tilde{\lambda}}^{b_{\tilde{\lambda},\Upsilon} - \tau} \quad \text{and} \quad \eta(u) = u, \text{ for } u \notin \phi_{\tilde{\lambda}}^{-1}([b_{\tilde{\lambda},\Upsilon} - 2\tau, b_{\tilde{\lambda},\Upsilon} + 2\tau]).$$

Then, by (6.19),

$$\eta(\gamma_0(\mathbf{t})) = \gamma_0(\mathbf{t}), \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

Now, using the definition of $b_{\tilde{\lambda}, \Upsilon}$, there exists $\gamma_* \in \Gamma_*$ satisfying

$$\max_{\mathbf{t} \in [1/R^2, 1]^l} \phi_{\tilde{\lambda}}(\gamma_*(\mathbf{t})) < b_{\tilde{\lambda}, \Upsilon} + \tau. \quad (6.20)$$

Defining

$$\tilde{\gamma}(\mathbf{t}) = \eta(\gamma_*(\mathbf{t})), \mathbf{t} \in [1/R^2, 1]^l,$$

due to (6.20), we obtain

$$\phi_{\tilde{\lambda}}(\tilde{\gamma}(\mathbf{t})) \leq b_{\tilde{\lambda}, \Upsilon} - \tau, \forall \mathbf{t} \in [1/R^2, 1]^l.$$

But since $\tilde{\gamma} \in \Gamma_*$, we deduce

$$b_{\tilde{\lambda}, \Upsilon} \leq \max_{\mathbf{t} \in [1/R^2, 1]^l} \phi_{\tilde{\lambda}}(\tilde{\gamma}(\mathbf{t})) \leq b_{\tilde{\lambda}, \Upsilon} - \tau,$$

a contradiction. So, $\tilde{\lambda} < \lambda_1$. ■

7 The proof of the main theorem

To prove Theorem 1.1, we need to find nonnegative solutions u_λ for large values of λ , which converges to a least energy solution in each Ω_j ($j \in \Upsilon$) and to 0 in Ω_Υ^c as $\lambda \rightarrow \infty$. To this end, we will show two propositions which together with the Propositions 4.1 and 5.1 will imply that Theorem 1.1 holds.

Henceforth, we denote by

$$r = R^{p_+} \sum_{j=1}^l \left(\frac{1}{p_+} - \frac{1}{\theta} \right)^{-1} c_j, \quad \mathcal{B}_r^\lambda = \{u \in E_\lambda; \varrho_\lambda(u) \leq r\}$$

and

$$\phi_\lambda^{c_\Upsilon} = \{u \in E_\lambda; \phi_\lambda(u) \leq c_\Upsilon\}.$$

Moreover, for small values of μ ,

$$\mathcal{A}_\mu^\lambda = \{u \in \mathcal{B}_r^\lambda; \varrho_{\lambda, \mathbb{R}^N \setminus \Omega_\Upsilon}(u) \leq \mu, |\phi_{\lambda, j}(u) - c_j| \leq \mu, \forall j \in \Upsilon\}.$$

We observe that

$$w = \sum_{j=1}^l w_j \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon},$$

showing that $\mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon} \neq \emptyset$. Fixing

$$0 < \mu < \frac{1}{4} \min_{j \in \Upsilon} c_j, \quad (7.21)$$

we have the following uniform estimate of $\|\phi'_\lambda(u)\|$ on the region $(\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda) \cap \phi_\lambda^{c_\Upsilon}$.

Proposition 7.1 *Let $\mu > 0$ satisfying (7.21). Then, there exist $\Lambda_* \geq 1$ and $\sigma_0 > 0$ independent of λ such that*

$$\|\phi'_\lambda(u)\| \geq \sigma_0, \text{ for } \lambda \geq \Lambda_* \text{ and all } u \in (\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda) \cap \phi_\lambda^{c_\Upsilon}. \quad (7.22)$$

Proof. We assume that there exist $\lambda_n \rightarrow \infty$ and $u_n \in (\mathcal{A}_{2\mu}^{\lambda_n} \setminus \mathcal{A}_\mu^{\lambda_n}) \cap \phi_{\lambda_n}^{c_\Upsilon}$ such that

$$\|\phi'_{\lambda_n}(u_n)\| \rightarrow 0.$$

Since $u_n \in \mathcal{A}_{2\mu}^{\lambda_n}$, this implies $(\varrho_{\lambda_n}(u_n))$ is a bounded sequence and, consequently, it follows that $(\phi_{\lambda_n}(u_n))$ is also bounded. Thus, passing a subsequence if necessary, we can assume $\phi_{\lambda_n}(u_n)$ converges. Thus, from Proposition 4.1, there exists $0 \leq u \in W_0^{1,p(x)}(\Omega_\Upsilon)$ such that $u|_{\Omega_j}$, $j \in \Upsilon$, is a solution for (P_j) ,

$$\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_n) \rightarrow 0 \text{ and } \phi_{\lambda_n, j}(u_n) \rightarrow I_j(u).$$

We know that c_j is the least energy level for I_j . So, if $u|_{\Omega_j} \neq 0$, then $I_j(u) \geq c_j$. But since $\phi_{\lambda_n}(u_n) \leq c_\Upsilon$, we must analyze the following possibilities:

- (i) $I_j(u) = c_j$, $\forall j \in \Upsilon$;
- (ii) $I_{j_0}(u) = 0$, for some $j_0 \in \Upsilon$.

If (i) occurs, then for n large, it holds

$$\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_n) \leq \mu \text{ and } |\phi_{\lambda_n, j}(u_n) - c_j| \leq \mu, \forall j \in \Upsilon.$$

So $u_n \in \mathcal{A}_\mu^{\lambda_n}$, a contradiction.

If (ii) occurs, then

$$|\phi_{\lambda_n, j_0}(u_n) - c_{j_0}| \rightarrow c_{j_0} > 4\mu,$$

which is a contradiction with the fact that $u_n \in \mathcal{A}_{2\mu}^{\lambda_n}$. Thus, we have completed the proof. ■

Proposition 7.2 *Let $\mu > 0$ satisfying (7.21) and $\Lambda_* \geq 1$ given in the previous proposition. Then, for $\lambda \geq \Lambda_*$, there exists a solution u_λ of (A_λ) such that $u_\lambda \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{cr}$.*

Proof. Let $\lambda \geq \Lambda_*$. Assume that there are no critical points of ϕ_λ in $\mathcal{A}_\mu^\lambda \cap \phi_\lambda^{cr}$. Since ϕ_λ is a (PS) functional, there exists a constant $d_\lambda > 0$ such that

$$\|\phi'_\lambda(u)\| \geq d_\lambda, \text{ for all } u \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{cr}.$$

From Proposition 7.1 we have

$$\|\phi'_\lambda(u)\| \geq \sigma_0, \text{ for all } u \in (\mathcal{A}_{2\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda) \cap \phi_\lambda^{cr},$$

where $\sigma_0 > 0$ does not depend on λ . In what follows, $\Psi: E_\lambda \rightarrow \mathbb{R}$ is a continuous functional verifying

$$\Psi(u) = 1, \text{ for } u \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda, \quad \Psi(u) = 0, \text{ for } u \notin \mathcal{A}_{2\mu}^\lambda \text{ and } 0 \leq \Psi(u) \leq 1, \quad \forall u \in E_\lambda.$$

We also consider $H: \phi_\lambda^{cr} \rightarrow E_\lambda$ given by

$$H(u) = \begin{cases} -\Psi(u) \|Y(u)\|^{-1} Y(u), & \text{for } u \in \mathcal{A}_{2\mu}^\lambda, \\ 0, & \text{for } u \notin \mathcal{A}_{2\mu}^\lambda, \end{cases}$$

where Y is a pseudo-gradient vector field for Φ_λ on $\mathcal{K} = \{u \in E_\lambda; \phi'_\lambda(u) \neq 0\}$. Observe that H is well defined, once $\phi'_\lambda(u) \neq 0$, for $u \in \mathcal{A}_{2\mu}^\lambda \cap \phi_\lambda^{cr}$. The inequality

$$\|H(u)\| \leq 1, \quad \forall \lambda \geq \Lambda_* \text{ and } u \in \phi_\lambda^{cr},$$

guarantees that the deformation flow $\eta: [0, \infty) \times \phi_\lambda^{cr} \rightarrow \phi_\lambda^{cr}$ defined by

$$\frac{d\eta}{dt} = H(\eta), \quad \eta(0, u) = u \in \phi_\lambda^{cr}$$

verifies

$$\frac{d}{dt} \phi_\lambda(\eta(t, u)) \leq -\frac{1}{2} \Psi(\eta(t, u)) \|\phi'_\lambda(\eta(t, u))\| \leq 0, \quad (7.23)$$

$$\left\| \frac{d\eta}{dt} \right\|_\lambda = \|H(\eta)\|_\lambda \leq 1 \quad (7.24)$$

and

$$\eta(t, u) = u \text{ for all } t \geq 0 \text{ and } u \in \phi_\lambda^{cr} \setminus \mathcal{A}_{2\mu}^\lambda. \quad (7.25)$$

We study now two paths, which are relevant for what follows:

- The path $\mathbf{t} \mapsto \eta(t, \gamma_0(\mathbf{t}))$, where $\mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l$.

The definition of γ_0 combined with the condition on μ gives

$$\gamma_0(\mathbf{t}) \notin \mathcal{A}_{2\mu}^\lambda, \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

Since

$$\phi_\lambda(\gamma_0(\mathbf{t})) < c_\Upsilon, \forall \mathbf{t} \in \partial[1/R^2, 1]^l,$$

from (7.25), it follows that

$$\eta(t, \gamma_0(\mathbf{t})) = \gamma_0(\mathbf{t}), \forall \mathbf{t} \in \partial[1/R^2, 1]^l.$$

So, $\eta(t, \gamma_0(\mathbf{t})) \in \Gamma_*$, for each $t \geq 0$.

- The path $\mathbf{t} \mapsto \gamma_0(\mathbf{t})$, where $\mathbf{t} = (t_1, \dots, t_l) \in [1/R^2, 1]^l$.

We observe that

$$\text{supp}(\gamma_0(\mathbf{t})) \subset \overline{\Omega_\Upsilon}$$

and

$$\phi_\lambda(\gamma_0(\mathbf{t})) \text{ does not depend on } \lambda \geq 1,$$

for all $\mathbf{t} \in [1/R^2, 1]^l$. Moreover,

$$\phi_\lambda(\gamma_0(\mathbf{t})) \leq c_\Upsilon, \forall \mathbf{t} \in [1/R^2, 1]^l$$

and

$$\phi_\lambda(\gamma_0(\mathbf{t})) = c_\Upsilon \text{ if, and only if, } t_j = \frac{1}{R}, \forall j \in \Upsilon.$$

Therefore

$$m_0 = \sup \{ \phi_\lambda(u) ; u \in \gamma_0([1/R^2, 1]^l) \setminus A_\mu^\lambda \}$$

is independent of λ and $m_0 < c_\Upsilon$. Now, observing that there exists $K_* > 0$ such that

$$|\phi_{\lambda,j}(u) - \phi_{\lambda,j}(v)| \leq K_* \|u - v\|_{\lambda, \Omega_j'}, \forall u, v \in \mathcal{B}_r^\lambda \text{ and } \forall j \in \Upsilon,$$

we derive

$$\max_{\mathbf{t} \in [1/R^2, 1]^l} \phi_\lambda(\eta(T, \gamma_0(\mathbf{t}))) \leq \max \left\{ m_0, c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu \right\}, \quad (7.26)$$

for $T > 0$ large.

In fact, writing $u = \gamma_0(\mathbf{t})$, $\mathbf{t} \in [1/R^2, 1]^l$, if $u \notin A_\mu^\lambda$, from (7.23),

$$\phi_\lambda(\eta(t, u)) \leq \phi_\lambda(u) \leq m_0, \forall t \geq 0,$$

and we have nothing more to do. We assume then $u \in A_\mu^\lambda$ and set

$$\tilde{\eta}(t) = \eta(t, u), \quad \tilde{d}_\lambda = \min \{d_\lambda, \sigma_0\} \quad \text{and} \quad T = \frac{\sigma_0 \mu}{K_* \tilde{d}_\lambda}.$$

Now, we will analyze the ensuing cases:

Case 1: $\tilde{\eta}(t) \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda, \forall t \in [0, T]$.

Case 2: $\tilde{\eta}(t_0) \in \partial \mathcal{A}_{\frac{3}{2}\mu}^\lambda$, for some $t_0 \in [0, T]$.

Analysis of Case 1

In this case, we have $\Psi(\tilde{\eta}(t)) = 1$ and $\|\phi'_\lambda(\tilde{\eta}(t))\| \geq \tilde{d}_\lambda$ for all $t \in [0, T]$. Hence, from (7.23),

$$\phi_\lambda(\tilde{\eta}(T)) = \phi_\lambda(u) + \int_0^T \frac{d}{ds} \phi_\lambda(\tilde{\eta}(s)) ds \leq c_\Upsilon - \frac{1}{2} \int_0^T \tilde{d}_\lambda ds,$$

that is,

$$\phi_\lambda(\tilde{\eta}(T)) \leq c_\Upsilon - \frac{1}{2} \tilde{d}_\lambda T = c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu,$$

showing (7.26).

Analysis of Case 2

In this case, there exist $0 \leq t_1 \leq t_2 \leq T$ satisfying

$$\begin{aligned} \tilde{\eta}(t_1) &\in \partial \mathcal{A}_\mu^\lambda, \\ \tilde{\eta}(t_2) &\in \partial \mathcal{A}_{\frac{3}{2}\mu}^\lambda, \end{aligned}$$

and

$$\tilde{\eta}(t) \in \mathcal{A}_{\frac{3}{2}\mu}^\lambda \setminus \mathcal{A}_\mu^\lambda, \quad \forall t \in (t_1, t_2].$$

We claim that

$$\|\tilde{\eta}(t_2) - \tilde{\eta}(t_1)\| \geq \frac{1}{2K_*} \mu.$$

Setting $w_1 = \tilde{\eta}(t_1)$ and $w_2 = \tilde{\eta}(t_2)$, we get

$$\varrho_{\lambda, \mathbb{R}^N \setminus \Omega_\Upsilon}(w_2) = \frac{3}{2} \mu \quad \text{or} \quad |\phi_{\lambda, j_0}(w_2) - c_{j_0}| = \frac{3}{2} \mu,$$

for some $j_0 \in \Upsilon$. We analyse the latter situation, once that the other one follows the same reasoning. From the definition of \mathcal{A}_μ^λ ,

$$|\phi_{\lambda,j_0}(w_1) - c_{j_0}| \leq \mu,$$

consequently,

$$\|w_2 - w_1\| \geq \frac{1}{K_*} |\phi_{\lambda,j_0}(w_2) - \phi_{\lambda,j_0}(w_1)| \geq \frac{1}{2K_*} \mu.$$

Then, by mean value theorem, $t_2 - t_1 \geq \frac{1}{2K_*} \mu$ and, this form,

$$\phi_\lambda(\tilde{\eta}(T)) \leq \phi_\lambda(u) - \int_0^T \Psi(\tilde{\eta}(s)) \|\phi'_\lambda(\tilde{\eta}(s))\| ds$$

implying

$$\phi_\lambda(\tilde{\eta}(T)) \leq c_\Upsilon - \int_{t_1}^{t_2} \sigma_0 ds = c_\Upsilon - \sigma_0(t_2 - t_1) \leq c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu,$$

which proves 7.26. Fixing $\hat{\eta}(t_1, \dots, t_l) = \eta(T, \gamma_0(t_1, \dots, t_l))$, we have that $\hat{\eta} \in \Gamma_*$ and, hence,

$$b_{\lambda,\Gamma} \leq \max_{(t_1, \dots, t_l) \in [1/R^2, 1]} \phi_\lambda(\hat{\eta}(t_1, \dots, t_l)) \leq \max \left\{ m_0, c_\Upsilon - \frac{1}{2K_*} \sigma_0 \mu \right\} < c_\Upsilon,$$

which contradicts the fact that $b_{\lambda,\Gamma} \rightarrow c_\Upsilon$. ■

Proof of Theorem 1.1. According Proposition 7.2, for μ satisfying (7.21) and $\Lambda_* \geq 1$, there exists a solution u_λ for (A_λ) such that $u_\lambda \in \mathcal{A}_\mu^\lambda \cap \phi_\lambda^{c_\Upsilon}$, for all $\lambda \geq \Lambda_*$.

Claim: There are $\lambda_0 \geq \Lambda_*$ and $\mu_0 > 0$ small enough, such that u_λ is a solution for (P_λ) for $\lambda \geq \lambda_0$ and $\mu \in (0, \mu_0)$.

Indeed, assume by contradiction that there are $\lambda_n \rightarrow \infty$ and $\mu_n \rightarrow 0$, such that (u_{λ_n}) is not a solution for (P_{λ_n}) . From Proposition 7.2, the sequence (u_{λ_n}) verifies:

- (a) $\phi'_{\lambda_n}(u_{\lambda_n}) = 0, \forall n \in \mathbb{N}$;
- (b) $\varrho_{\lambda_n, \mathbb{R}^N \setminus \Omega_\Upsilon}(u_{\lambda_n}) \rightarrow 0$;
- (c) $\phi_{\lambda_n, j}(u_{\lambda_n}) \rightarrow c_j, \forall j \in \Upsilon$.

The item (b) ensures we can use Proposition 5.1 to deduce u_{λ_n} is a solution for (P_{λ_n}) , for large values of n , which is a contradiction, showing this way the claim.

Now, our goal is to prove the second part of the theorem. To this end, let (u_{λ_n}) be a sequence verifying the above limits. Since $\phi_{\lambda_n}(u_{\lambda_n})$ is bounded, passing a subsequence, we obtain that $\phi_{\lambda_n}(u_{\lambda_n}) \rightarrow c$. This way, using Proposition 4.1 combined with item (c), we derive u_{λ_n} converges in $W^{1,p(x)}(\mathbb{R}^N)$ to a function $u \in W^{1,p(x)}(\mathbb{R}^N)$, which satisfies $u = 0$ outside Ω_Υ and $u|_{\Omega_j}$, $j \in \Upsilon$, is a least energy solution for

$$\begin{cases} -\Delta_{p(x)}u + Z(x)u = f(u), & \text{in } \Omega_j, \\ u \in W_0^{1,p(x)}(\Omega_j), u \geq 0, & \text{in } \Omega_j. \end{cases}$$

■

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